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Non-triviality of some secondary characteristic classes of foliations

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§1 Introduction

Bott and Haefliger [3] and Bernstein and Rosenfeld [1] defined characteristic classes of foliations with various geometric structures. In particular, for codimension n smooth foliations with trivialized normal bundles, they consider certain differential graded algebra W_n and construct a homomorphism

$$\varphi : H^*(W_n) \longrightarrow H^*(B\overline{\Gamma}_n; \mathbb{R})$$

where $B\overline{\Gamma}_n$ is the fibre of the natural map $B\Gamma_n \longrightarrow BGL_n\mathbb{R}$.

The purpose of this note is to show that φ-images of some elements of $H^*(W_n)$ are non-trivial. More precisely, we shall show the following.

Theorem 1. $\varphi(h_1 h_{i_1} \cdots h_{i_k} c^{\alpha})$ is non-trivial if $1 < i_1 < \cdots < i_k \leqq n$ and c^{α} is a monomial of $c_1, \ldots, c_n$ of degree $2n$.

For codimension n smooth foliations, we have characteristic classes,

$$\varphi : H^*(WO_n) \longrightarrow H^*(B\Gamma_n; \mathbb{R}).$$

An immediate consequence of Theorem 1 is the following.

Corollary 2. $\varphi(h_1 h_{i_1} \cdots h_{i_k} c^{\alpha})$ is non-trivial if $1 < i_1 < \cdots < i_k \leqq n$, i_j: odd for all j and c^{α} is a monomial of $c_1, \ldots, c_n$ of degree $2n$.

The proof of Theorem 1 will be given by computing the secondary classes of a particular foliation, namely a homogeneous codimension n foliation on $SL_{n+1}\mathbb{R}$. We can also consider a codimension n holomorphic foliation on $SL_{n+1}\mathbb{C}$. For this type of foliations, we have secondary classes.

$$\varphi^{\mathbb{C}} : H^*(W_n^{\mathbb{C}}) \longrightarrow H^*(B\bar{\Gamma}_n^{\mathbb{C}}; \mathbb{C}).$$

The same proof as that of Theorem 1 shows

Theorem 3. $\varphi^{\mathbb{C}}(h_1 h_{i_1} \cdots h_{i_k} c^{\alpha})$ is non-trivial if $1 < i_1 < \cdots < i_k \leqq n$ and c^{α} is a monomial of $c_1, \ldots, c_n$ of degree 2n.

Non-triviality of $\varphi(h_1 h_2 \cdots h_n c_1^n)$ has been already mentioned in Bernstein and Rosenfeld [2] and Corollary 2 is proved by Kamber and Tondeur in a recently published Lecture Notes [5] as an application of their systematic theory. Our method is more elementary and computational. Therefore we shall only summarize the outline of the proofs. Finally, we mention that the secondary classes of homogeneous foliations on $SO(n+1, 1)/SO(n)$ has been calculated by Yamato [6].

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§ 2 The curvature form

Let $H = \{(x_{ij}) \in SL_{n+1}\mathbb{R};\ x_{21} = \cdots = x_{n+1\,1} = 0\}$ be a subgroup of $SL_{n+1}\mathbb{R}$. We define a codimension n foliation $\mathcal{F}$ on $SL_{n+1}\mathbb{R}$ by the right cosets $\{gH\}_{g \in SL_{n+1}\mathbb{R}}$.

Let $s\ell_{n+1}\mathbb{R}$ be the Lie algebra of $SL_{n+1}\mathbb{R}$. Then $\{e_{ij}, f_k\}_{\substack{i,j=1,\ldots,n+1,\ i\neq j \\ k=2,\ldots,n+1}}$ forms a basis of $s\ell_{n+1}\mathbb{R}$, where

$$e_{ij} = \begin{array}{cc} & j \\ i & \begin{pmatrix} \cdots\cdots & 1 \end{pmatrix} \end{array}, \quad f_k = \begin{array}{c} k \\ \begin{bmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} \end{array}.$$

Let $\{e_{ij}^*, f_k^*\}$ be the dual basis of $(s\ell_{n+1}\mathbb{R})^*$. If we consider an element of $(s\ell_{n+1}\mathbb{R})^*$ to be a left invariant 1-form on $SL_{n+1}\mathbb{R}$, then it is clear that $\mathcal{F}$ is defined by the equations,

$$e^*_{i1} = 0 \quad i = 2, \ldots, n+1.$$

Direct computation shows

$$de^*_{i1} = -\sum_{j\neq 1,i} e^*_{ij} \wedge e^*_{j1} + \sum_{j=2}^{n+1} f^*_j \wedge e^*_{i1} + f^*_i \wedge e^*_{i1}.$$

Therefore the connection form θ associated to the Bott connection of normal bundle of $\mathcal{F}$ is given as follows.

$$(1) \quad \theta = \begin{bmatrix} -\left(\sum_{j=2}^{n+1} f^*_j\right) - f^*_2 & e^*_{23} & \cdots\cdots & e^*_{2\ n+1} \\ e^*_{32} & -\left(\sum_{j=2}^{n+1} f^*_j\right) - f^*_3 & \cdots\cdots & e^*_{3\ n+1} \\ \vdots & \vdots & & \vdots \\ e^*_{n+1\ 1} & e^*_{n+1\ 2} & \cdots\cdots & -\left(\sum_{j=2}^{n+1} f^*_j\right) - f^*_{n+1} \end{bmatrix}$$

Again direct calculation yields

$$d\theta = ((d\theta)_{ij}), \quad \text{where}$$

$$(2) \qquad (d\theta)_{ii} = \sum_{k=2}^{n+1} e^*_{1k} \wedge e^*_{k1} + \sum_{j\neq i+1} e^*_{j\ i+1} \wedge e^*_{i+1\ j}$$

$$(d\theta)_{ij} = \sum_{k\neq i+1,j+1} e^*_{k\ j+1} \wedge e^*_{i+1\ k} + (f^*_{i+1} - f^*_{j+1}) \wedge e^*_{i+1\ j+1} \quad (i\neq j).$$

Moreover we have

$$\theta^2 = ((\theta^2))_{ij}), \quad \text{where}$$

$$(3) \qquad (\theta^2)_{ii} = \sum_{k\neq 1,i+1} e^*_{i+1\ k} \wedge e^*_{k\ i+1}$$

$$(\theta^2)_{ij} = \sum_{k\neq 1,i+1,j+1} e^*_{i+1\ k} \wedge e^*_{k\ j+1} - (f^*_{i+1} - f^*_{j+1}) \wedge e^*_{i+1\ j+1} \quad (i\neq j).$$

Now the curvature matrix k is given by

$$k = d\theta + \theta^2.$$

Therefore, from (2) and (3), we have

$$(4)\quad k=\begin{bmatrix} \sum_{k\neq 1}(e^*_{1k}\wedge e^*_{k1})+e^*_{12}\wedge e^*_{21} & e^*_{13}\wedge e^*_{21} & \cdots\cdots, & e^*_{1\,n+1}\wedge e^*_{21} \\ e^*_{12}\wedge e^*_{31} & \sum_{k\neq 1}(e^*_{1k}\wedge e^*_{k1})+e^*_{13}\wedge e^*_{31} & \cdots\cdots & e^*_{1\,n+1}\wedge e^*_{31} \\ \vdots & \vdots & & \vdots \\ e^*_{12}\wedge e^*_{n+1\,1} & e^*_{13}\wedge e^*_{n+1\,2} & \cdots\cdots & \sum_{k\neq 1}(e^*_{1k}\wedge e^*_{k1})+e^*_{1\,n+1}\wedge e^*_{n+1\,1} \end{bmatrix}$$

The c_i form of $\mathcal{F}$ is defined to be the degree $2i$ part of $\det(E + \frac{1}{2\pi} k)$. Now let c^α be a monomial of $c_1, \ldots, c_n$ of degree $2n$. Then from (4), we conclude

$$(5)\quad c^\alpha = K_\alpha\, e^*_{12}\wedge e^*_{21} \wedge \cdots\cdots \wedge e^*_{1\,n+1}\wedge e^*_{n+1\,1}$$

where K_α is a non-zero constant.

§ 3 Cohomology of $s\ell_{n+1}\mathbb{R}$

In this section, let us recall the cohomology of $s\ell_{n+1}\mathbb{R}$. Since the compact form of $s\ell_{n+1}\mathbb{C}$ is su_{n+1}, we have isomorphisms

$$(6)\qquad H^*(s\ell_{n+1}\mathbb{R}) \cong H^*(su_{n+1}) \cong H^*(SU_{n+1})$$

$$(7)\qquad \begin{aligned} H^*(s\ell_{n+1}\mathbb{R}, s\ell_n\mathbb{R}) &\cong H^*(su_{n+1}, su_n) \\ &\cong H^*(SU_{n+1}/SU_n) \\ &= H^*(S^{2n+1}) \end{aligned}$$

where we consider $s\ell_n\mathbb{R}$ to be a subalgebra of $s\ell_{n+1}\mathbb{R}$;

$$i : s\ell_n\mathbb{R} \longrightarrow s\ell_{n+1}\mathbb{R}$$

$$A \longrightarrow \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & A & \\ 0 & & \end{bmatrix}$$

Since the fibration $SU_n \longrightarrow SU_{n+1} \longrightarrow S^{2n+1}$ is rational homotopy theoretically trivial, we have a rational homotopy equivalence

$$SU_{n+1} \underset{\mathbb{Q}}{\sim} S^3 \times S^5 \times \dots \times S^{2n+1}.$$

Therefore we have

(8) $$H^*(\mathfrak{sl}_{n+1}\mathbb{R}) \cong H^*(S^3 \times S^5 \times \dots \times S^{2n+1}, \mathbb{R}).$$

Using the theory of harmonic integrals (Hodge [4]) and the isomorphism (6), we can construct a closed ℓ-form $\Omega_\ell \in A^\ell(\mathfrak{sl}_{n+1}\mathbb{R})$ ($\ell = 3, 5, \dots, 2n+1$), whose image under the isomorphism (8) evaluates non trivially on the fundamental cycle of S^ℓ, as follows.

Let $(x_{ij}) \in SL_{n+1}\mathbb{R}$ and let us write $(X_{ij}) = (x_{ij})^{-1}$. We define a 1-form $\zeta^i{}_k$ by

$$\zeta^i{}_k = \sum_{j=1}^{n+1} X_{ij}dx_{jk}.$$

Then $\zeta^i{}_k$ can be seen to be left invariant. In fact we can show

$$\zeta^i{}_k = e^*_{ik} \qquad (i \neq k)$$

(9) $$\zeta^1{}_1 = f^*_2 + \dots + f^*_{n+1}$$

$$\zeta^i{}_i = -f^*_i \qquad (i = 2, \dots, n+1).$$

Now Ω_ℓ is defined by

(10) $$\Omega_\ell = \sum \zeta^{i_1}{}_{i_2}\zeta^{i_2}{}_{i_3} \cdots \zeta^{i_\ell}{}_{i_1}.$$

From this definition, it is clear that Ω_ℓ is natural under the inclusion $i : \mathfrak{sl}_n\mathbb{R} \longrightarrow \mathfrak{sl}_{n+1}\mathbb{R}$. Thus we may write $i^*\Omega_\ell = \Omega_\ell$ ($\ell = 3, 5, \dots, 2n-1$). Next we define a closed $(2n+1)$-form $\bar{\Omega}_{2n+1}$ by

(11) $$\bar{\Omega}_{2n+1} = (f^*_2 + \dots + f^*_{n+1}) \wedge e^*_{12} \wedge e^*_{21} \wedge \dots \wedge e^*_{1\,n+1} \wedge e^*_{n+1\,1}.$$

Then it is easy to see that $\bar{\Omega}_{2n+1}$ is a generator of $A^{2n+1}(\mathfrak{sl}_{n+1}\mathbb{R}, \mathfrak{sl}_n\mathbb{R})$. The most important fact about these classes is the following.

(12) $\quad \Omega_3 \Omega_5 \cdots \Omega_{2n-1} \overline{\Omega}_{2n+1}$ is a volume form on $s\ell_{n+1}\mathbb{R}$.

For the details of these results, see [4].

§4 The h_i form

Now let us calculate the h_i form. On the manifold $I \times SL_{n+1}\mathbb{R}$, we consider 1-form $\tilde{\theta} = t\theta$ and the corresponding curvature form

(13) $$\tilde{K} = d\tilde{\theta} + \tilde{\theta}^2 = dt\,\theta + td\theta + t^2\theta^2.$$

Let $\tilde{c}_i$ be the degree 2i part of $\det(E + \frac{1}{2\pi}\tilde{K})$. If we write $\tilde{c}_i = dt\, a_i + b_i$, then h_i is defined by

$$h_i = \int_0^1 a_i dt.$$

First we determine h_1. Since $\tilde{c}_1 = -\frac{n+1}{2\pi}dt(f_2^* + \cdots + f_{n+1}^*)$. We have

$$h_1 = -\frac{n+1}{2\pi}(f_2^* + \cdots + f_{n+1}^*).$$

Therefore we have

(14) $$\begin{aligned} h_1 c^\alpha &= -\frac{n+1}{2\pi}K_\alpha(f_2^* + \cdots + f_{n+1}^*) \wedge e_{12}^* \wedge e_{21}^* \wedge \cdots \wedge e_{1\,n+1}^* \wedge e_{n+1\,1}^* \\ &= -\frac{n+1}{2\pi}K_\alpha \overline{\Omega}_{2n+1} \qquad \text{(see (11))}. \end{aligned}$$

Next we consider the h_i form for $i = 2, 3, .., n$. Recall that $i : s\ell_n\mathbb{R} \longrightarrow s\ell_{n+1}\mathbb{R}$ is the natural inclusion. Then we have

Lemma 4. $i^*(h_i) = \overline{T}_i\Omega_{2i-1}$ + exact form. ($\overline{T}_i$ is a non-zero constant).

Proof. By (2) and (3), we have

(15) $$i^*(d\theta) = -i^*(\theta^2).$$

Therefore from (13), we obtain

(16) $$i^*(\tilde{K}) = dt\, i^*(\theta) + (t^2 - t)i^*(\theta^2).$$

By induction, we have

(17) $$i^*(\tilde{K}^i) = i(t^2 - t)^{i-1} dt\, i^*(\theta^{2i-1}) + (t^2 - t)^i i^*(\theta^{2i}).$$

Now let us define a class $\widetilde{\Sigma}_i$ to be $\mathrm{Tr}(\frac{1}{2\pi}\widetilde{k})^i$. Then we have

(18) $$i^*(\widetilde{\Sigma}_i) = (\tfrac{1}{2\pi})^i i(t^2-t)^{i-1}dt\ \mathrm{Tr}\ i^*(\theta^{2i-1}) + (\tfrac{1}{2\pi})^i (t^2-t)^i \mathrm{Tr}\ i^*(\theta^{2i}).$$

As is well-known, $\widetilde{c}_i$ can be written uniquely in terms of $\widetilde{\Sigma}_i$ $(i = 1, \ldots, n)$,

(19) $$\widetilde{c}_i = (-1)^{i+1}\tfrac{1}{i}\widetilde{\Sigma}_i + \text{decomposable terms}.$$

Combining (18) and (19), we obtain

(20) $$i^*(\widetilde{c}_i) = (-1)^{i+1}(\tfrac{1}{2\pi})^i (t^2-t)^{i-1}dt\ \mathrm{Tr}\ i^*(\theta^{2i-1}) + (-1)^{i+1}(\tfrac{1}{2\pi})^i \tfrac{1}{i}(t^2-t)^i \mathrm{Tr}\ i^*(\theta^{2i}) + \text{decomposable terms}.$$

Therefore

(21) $$i^*(h_i) = (-1)^{i+1}(\tfrac{1}{2\pi})^i T_i\ \mathrm{Tr}\ i^*(\theta^{2i-1}) + D$$

where $T_i = \int_0^1 (t^2-t)^{i-1}dt$ $(\neq 0)$ and D is a term contributed from the decomposable terms in (20). Clearly D is a linear combination of forms of the following type.

(22) $$\mathrm{Tr}\ i^*(\theta^{2i_1-1})\mathrm{Tr}\ i^*(\theta^{2i_2}) \cdots \mathrm{Tr}\ i^*(\theta^{2i_s}), \quad i_1+\cdots+i_s=i.$$

Now from (1), we have

(23) $$i^*(\theta) = \begin{pmatrix} f_2^*+\cdots+f_n^* & e_{12}^* & \cdots\cdots & e_{1n}^* \\ e_{21}^* & -f_2^* & \cdots\cdots & e_{2n}^* \\ & & & \\ e_{n1}^* & e_{n2}^* & \cdots\cdots & -f_n^* \end{pmatrix}.$$

Here f_k^* and e_{ij}^* should be understood to be elements of $(s\ell_n\mathbb{R})^*$. Combining (9), (10) and (23), we obtain

(24) $$\mathrm{Tr}\ i^*(\theta^{2i-1}) = \Omega_{2i-1} \in A^{2i-1}(s\ell_n\mathbb{R}).$$

Now recall that $i^*(\theta^2) = -i^*(d\theta)$ (see (16)) and Ω_{odd} is a closed form. Therefore the form (22) is an exact form. From (21) and (24) we have

(25) $\quad i^*(h_i) = (-1)^{i+1}(\frac{1}{2\pi})^i T_i \Omega_{2i-1}$ + exact form.

This proves Lemma 4.

§ 5 Proof of Theorem 1

Let $\Gamma \subset SL_{n+1}\mathbb{R}$ be a discrete subgroup such that $\Gamma\backslash SL_{n+1}\mathbb{R}$ is a compact manifold. Clearly the action of Γ on $SL_{n+1}\mathbb{R}$ preserves the foliation $\mathcal{F}$, hence we have a codimension n foliation on (n^2+2n)-dimensional manifold $\Gamma\backslash SL_{n+1}\mathbb{R}$. Since $H^*(s\ell_{n+1}\mathbb{R})$ admits the Poincaré duality (recall that $H^*(s\ell_{n+1}\mathbb{R}) \cong H^*(SU_{n+1}, \mathbb{R})$ as algebras), the natural map

$$H^*(s\ell_{n+1}\mathbb{R}) \longrightarrow H^*(\Gamma\backslash SL_{n+1}\mathbb{R}, \mathbb{R})$$

is injective. Thus to prove Theorem 1, it is enough to show that $[h_1 h_{i_1} \cdots h_{i_k} c^{\alpha}] \neq 0$, where $[\ \ ]$ denotes the cohomology class.

Now let $K = \mathrm{Ker}(i^* : A^*(s\ell_{n+1}\mathbb{R}) \longrightarrow A^*(s\ell_n\mathbb{R}))$. Then it is easy to see the following.

(26) K is the ideal of $A^*(s\ell_{n+1}\mathbb{R})$ generated by $(f_2^* + \cdots + f_{n+1}^*)$, $e_{12}^*, e_{21}^*, \ldots, e_{1\,n+1}^*, e_{n+1\,1}^*$.

Now since $i^*(h_{i_j}) = \bar{T}_{i_j}\Omega_{2i_j-1} + d\alpha_{i_j}$ for some α_{i_j} (see Lemma 4), we have

(27) $$h_{i_j} = \bar{T}_{i_j}\Omega_{2i_j-1} + d\bar{\alpha}_{i_j} + \beta_{i_j}$$

for some $\bar{\alpha}_{i_j} \in A^{2i_j-2}(s\ell_{n+1}\mathbb{R})$ and $\beta_{i_j} \in K$.

Now recall that $h_1 c^{\alpha} = -\frac{n+1}{2\pi}K_{\alpha}(f_2^* + \cdots + f_{n+1}^*)e_{12}^* \wedge e_{21}^* \wedge \cdots \wedge e_{1\,n+1}^* \wedge e_{n+1\,1}^*$ (see (14)). Therefore by (26), we have

(28) $$h_1 c^{\alpha}\beta_{i_j} = 0 \quad \text{for all } i_j.$$

Combining (14), (27) and (28), we conclude

$$(29) \quad h_1 h_{i_1} \cdots h_{i_k} c^{\alpha} = -\frac{n+1}{2\pi} K_{\alpha} \bar{T}_{i_1} \cdots \bar{T}_{i_k} \Omega_{2i_1-1} \cdots \Omega_{2i_k-1} \bar{\Omega}_{2n+1}$$

$$+ \text{ exact form.}$$

By the result of §3 (in particular (12)), cohomology class of the right hand side of (29) is different from zero. This proves Theorem 1.

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